SEMIGROUPS OF COMPOSITION OPERATORS ON HARDY SPACES OF THE HALF-PLANE

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Abstract. We identify the semigroups consisting of bounded composition operators on the Hardy spaces $H^p(\mathbb{U})$ of the upper half-plane. We show that any such semigroup is strongly continuous on $H^p(\mathbb{U})$ but not uniformly continuous and we identify the infinitesimal generator.

1. Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ denote the upper half of the complex plane. The Hardy space $H^p(\mathbb{U})$, $0 < p < \infty$, is the space of analytic functions $f : \mathbb{U} \to \mathbb{C}$ for which

$$\|f\|_p = \sup_{y>0} \Big(\int_{-\infty}^{\infty} |f(x+iy)|^p \, dx \Big)^{\frac{1}{p}} < \infty.$$

For $1 \leq p \leq \infty$, the spaces $H^p(\mathbb{U})$ are Banach spaces and $H^2(\mathbb{U})$ is a Hilbert space. Furthermore for $f \in H^p(\mathbb{U})$, $1 \leq p < \infty$, the limit $\lim_{y \to 0} f(x+iy)$ exists for almost every $x \in \mathbb{R}$ and we may define the boundary function on $\mathbb{R}$, again denoted by f, as

$$f(x) = \lim_{y \to 0} f(x+iy).$$

This function is p-integrable and

$$\|f\|_p^p = \int_{-\infty}^{\infty} |f(x)|^p \, dx.$$

For more details on Hardy spaces see [7], [11].

Let $\mathcal{H}(\mathbb{U})$ denote the space of all analytic functions on $\mathbb{U}$ and $\phi : \mathbb{U} \to \mathbb{U}$ be analytic. The composition operator induced by ϕ is defined by

$$C_\phi(f) = f \circ \phi, \quad f \in \mathcal{H}(\mathbb{U}).$$

If X is a linear subspace of $\mathcal{H}(\mathbb{U})$ which is a Banach space under a norm $\| \; \|_X$ we can consider the restriction of C_ϕ on X. The question arises whether C_ϕ acts as a bounded operator on X, that is if $f \circ \phi \in X$ for each $f \in X$ and if that is the case if $\|f \circ \phi\|_X \leq C \|f\|_X$ for a constant C. We will not consider this question here, but we mention that in contrast to the case of the Hardy spaces of the unit disc, there are self-maps ϕ of $\mathbb{U}$ which do not induce bounded composition operators on the Hardy spaces $H^p(\mathbb{U})$. More details for this will be presented in the next section.

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Suppose now that $\{\phi_t : t \geq 0\}$ is a one-parameter semigroup under composition of analytic self-maps of $\mathbb{U}$, that is:

(1) $\phi_0(z) \equiv z$, the identity map of $\mathbb{U}$.
(2) $\phi_{t+s} = \phi_t \circ \phi_s$ for $t, s \geq 0$.
(3) The map $(t,z) \to \phi_t(z)$ is jointly continuous on $[0, +\infty) \times \mathbb{U}$.

Then the induced maps

$$T_t(f) = f \circ \phi_t$$

form a semigroup of linear transformations on $\mathcal{H}(\mathbb{U})$. Semigroups of analytic functions on the half-plane and on the disc was first studied by E. Berkson and H. Porta in [2], where they also prove the strong continuity of the induced semigroups of composition operators on the Hardy spaces of the disc. Here we are concerned with analogous questions on the Hardy spaces of the half-plane.

Considering $\{T_t\}$ on a $H^p(\mathbb{U})$ space we show that if one $T_t,\ t > 0$ is a bounded operator then all T_t are bounded, giving also examples of unbounded semigroups $\{T_t\}$. Assuming further that $1 \leq p < \infty$ and each T_t is bounded on $H^p(\mathbb{U})$, we prove the strong continuity of $\{T_t\}$ on $H^p(\mathbb{U})$ and identify its infinitesimal generator.

A specific semigroup of composition operators was used in [1] to study the Cesàro operator and its adjoint on the Hardy spaces of the half-plane.

2. Preliminaries

2.1. **Composition operators on $H^p(\mathbb{U})$.** Recent results give characterizations of bounded composition operators on $H^p(\mathbb{U})$ spaces in terms of angular derivatives. Next we recall the necessary definitions and tools for them.

Let $f : \mathbb{U} \to \mathbb{C}$ be an analytic function. If $f(z) \to c$, where $c \in \mathbb{C} \cup \{\infty\}$, as $z = x + iy \to \infty$ through any sector

$$T_u(\infty) = \{x + iy \in \mathbb{U} : |x| < uy\}, \quad u > 0,$$

we say that c is the non-tangential limit of f at ∞ and we denote it by

$$\angle \lim_{z \to \infty} f(z).$$

Let $\phi : \mathbb{U} \to \mathbb{U}$ be an analytic function. The Julia-Carathéodory theorem for the upper half-plane (see [6, Exercise 2.3.10]) says that

$$\angle \lim_{z \to \infty} \frac{\phi(z)}{z} = \angle \lim_{z \to \infty} \phi'(z) = \inf_{z \in \mathbb{U}} \frac{\mathrm{Im}\phi(z)}{\mathrm{Im}z}.$$

From this it is clear that the limit

$$\phi'(\infty) := \angle \lim_{z \to \infty} \frac{\phi(z)}{z},$$

which we call it the angular derivative of ϕ at ∞, always exists and it belongs to $[0,\ +\infty)$.

We consider also the conjugate function ψ of ϕ on the disc $\mathbb{D} = \{w \in \mathbb{C} : |w| < 1\}$,

$$\psi = \gamma^{-1} \circ \phi \circ \gamma : \mathbb{D} \to \mathbb{D},$$

where $\gamma(w) = i\frac{1+w}{1-w}$, a conformal map from $\mathbb{D}$ onto $\mathbb{U}$, with inverse $\gamma^{-1}(z) = \frac{z-i}{z+i}$, $z \in \mathbb{U}$. Similar we denote the non-tangential limit of ψ at $\zeta \in \partial\mathbb{D}$, if this exists, by

$$\psi(\zeta) := \angle \lim_{w \to \zeta} \psi(w),$$

that is the limit of $\psi(w)$ as $w \to \zeta$ through any sector

$$S_a(\zeta) = \{w \in \mathbb{D} : |w - \zeta| < a(1 - |w|)\}, \quad a > 1.$$

In particular if $\psi(1) = 1$, we will use the angular derivative of ψ at 1,

$$\psi'(1) = \angle \lim_{w \to 1} \frac{1 - \psi(w)}{1 - w},$$

which is known (we see it also by Lemma 2.2) that always exists (it may be ∞).

Lemma 2.1. *Let $w \in \mathbb{D}$. Then $w \to 1$ non-tangentially if and only if $\gamma(w) \to \infty$ non-tangentially.*

Proof. It is clear that $w \to 1$ if and only if $\gamma(w) = i\frac{1+w}{1-w} \to \infty$. Let $a > 1$, $w \in S_a(1)$ and $z = \gamma(w) = x + iy$. Since

$$\begin{aligned} \frac{|w-1|}{1-|w|} = \frac{|\gamma^{-1}(z) - 1|}{1 - |\gamma^{-1}(z)|} &= \frac{2}{|z+i| - |z-i|} \\ &= \frac{2(|z+i| + |z-i|)}{|z+i|^2 - |z-i|^2} > \frac{4|x|}{(y+1)^2 - (y-1)^2} = \frac{|x|}{y} \end{aligned}$$

we get that $z \in T_a(\infty)$. Conversely let $u > 0$ and $z \in T_u(\infty)$ with $y > 1$. Then $\frac{|x|}{y} < u$ and it follows that

$$2(u+1) > \frac{2(|x| + y)}{y} = \frac{|x| + y + 1 + |x| + y - 1}{y} > \frac{|z+i| + |z-i|}{|z+i|^2 - |z-i|^2},$$

thus by the above we get that $w \in S_{4(u+1)}(1)$ and the conclusion follows. $\square$

Lemma 2.2. *Let $\phi : \mathbb{U} \to \mathbb{U}$ be analytic and ψ its conjugate on $\mathbb{D}$. If $\psi(1) = 1$, then*

$$\psi'(1) = \frac{1}{\phi'(\infty)}. \tag{2.1}$$

In particular, $\psi(1) = 1$ and $\psi'(1) < \infty$ if and only if $\phi'(\infty) > 0$.

Proof. We have

$$\begin{aligned} \psi'(1) = \angle \lim_{w \to 1} \frac{1 - \psi(w)}{1 - w} &= \angle \lim_{w \to 1} \frac{(1 - \psi(w))(1 + w)}{(1 + \psi(w))(1 - w)} = \angle \lim_{w \to 1} \frac{\gamma(w)}{\gamma(\psi(w))} \\ &= \angle \lim_{z \to \infty} \frac{z}{\phi(z)} = \frac{1}{\phi'(\infty)}. \end{aligned}$$

If $\phi'(\infty) > 0$, then by definition follows that $\angle \lim_{z \to \infty} \phi(z) = \infty$, so

$$\angle \lim_{w \to 1} \psi(w) = \angle \lim_{z \to \infty} \gamma^{-1}(\phi(z)) = 1$$

and furthermore $\psi'(1) < \infty$. $\square$

Now let $0 < p < \infty$ and $\phi : \mathbb{U} \to \mathbb{U}$ be analytic. V. Matache ([13, Theorem 15], [12]) showed that the induced composition operator $C_\phi : H^p(\mathbb{U}) \to H^p(\mathbb{U})$ is bounded if and only if the conjugate function ψ of ϕ has $\psi(1) = 1$ and $\psi'(1) < \infty$, i.e. if and only if $\phi'(\infty) > 0$. But the exact norm of C_ϕ was found by S. Elliott and M. Jury in [9] (see [9, Corollary 3.5 and Definition 2.4]) and is

$$\|C_\phi\| = \phi'(\infty)^{-\frac{1}{p}}. \tag{2.2}$$

Also an important role in the study of C_ϕ plays the Denjoy-Wolff point of ϕ. We recall that the Denjoy-Wolff theorem for analytic self-maps of $\mathbb{D}$ [6, Th. 2.51]

through the map γ asserts that if ϕ is not the identity or an elliptic automorphism, there is a point $d \in \overline{\mathbb{U}} = \mathbb{U} \cup \mathbb{R} \cup \{\infty\}$ such that the sequence of iterates $\phi_n \to d$ uniformly on compact subsets of $\mathbb{U}$. In the case of elliptic automorphism ϕ has a fixed point $d \in \mathbb{U}$. In both cases we call d the DW point of ϕ.

Corollary 2.3. *Let $0 < p < \infty$, $\phi : \mathbb{U} \to \mathbb{U}$ be analytic and C_ϕ the induced composition operator on $H^p(\mathbb{U})$. Then*

$$\|C_\phi\| \leq 1$$

if and only if the DW point of ϕ is ∞.

Proof. The DW point of ϕ is ∞ if and only if the DW point of its conjugate ψ is 1, which is equivalent to the conditions $\psi(1) = 1$ and $\psi'(1) \leq 1$ (see [15, Grand Iteration Theorem, p. 78]). By this and Lemma 2.2 the DW point of ϕ is ∞ if and only if $\phi'(\infty) \geq 1$, which from (2.2) is equivalent to $\|C_\phi\| \leq 1$. □

2.2. **Semigroups of analytic self-maps of** $\mathbb{U}$. Let $\{\phi_t : t \geq 0\}$ be a semigroup of analytic self-maps of $\mathbb{U}$. From [2], the analytic function $G : \mathbb{U} \to \mathbb{C}$ given by

$$G(z) = \lim_{t \to 0} \frac{\partial \phi_t(z)}{\partial t}$$

is the infinitesimal generator of $\{\phi_t\}$, characterizes $\{\phi_t\}$ uniquely and satisfies

$$G(\phi_t(z)) = \frac{\partial \phi_t(z)}{\partial t}, \quad z \in \mathbb{U}, \quad t \geq 0.$$

Suppose $\{\phi_t\}$ is not trivial, where $\{\phi_t\}$ is called trivial if $\phi_t(z) \equiv z$ for all t, then it turns out by [2, Theorem 2.6] that all ϕ_t, $t > 0$, have a common DW point d. Moreover if $d \neq \infty$, then $G(z)$ has the unique representation

$$G(z) = F(z)(z - \overline{d})(z - d), \tag{2.3}$$

where $F : \mathbb{U} \to \mathbb{C}$ is analytic, $F \not\equiv 0$, with $\mathrm{Im}F \geq 0$ on $\mathbb{U}$, while if $d = \infty$, then $\mathrm{Im}G \geq 0$ and $G \not\equiv 0$ on $\mathbb{U}$. The trivial semigroup has generator $G \equiv 0$.

Likewise let $\widetilde{G}$ be the infinitesimal generator of the conjugate semigroup $\{\psi_t\}$ of $\{\phi_t\}$. Then $\widetilde{G}(\psi_t(w)) = \frac{\partial \psi_t(w)}{\partial t}$, $w \in \mathbb{D}$, and for each $z \in \mathbb{U}$,

$$\widetilde{G}(\psi_t(\gamma^{-1}(z))) = \frac{\partial \psi_t(\gamma^{-1}(z))}{\partial t} = \frac{\partial \gamma^{-1}(\phi_t(z))}{\partial t} = \frac{2i}{(\phi_t(z) + i)^2} \frac{\partial \phi_t(z)}{\partial t}.$$

Hence letting t tends to 0 we get

$$\widetilde{G}(\gamma^{-1}(z)) = \frac{2i}{(z + i)^2} G(z). \tag{2.4}$$

Proposition 2.4. *Let $\{\phi_t : t \geq 0\}$ be a semigroup of analytic self-maps of $\mathbb{U}$ with DW point d. Then we can classify $\{\phi_t\}$ as follows.*
1) If $d \in \mathbb{U}$, then there is a unique univalent function $h : \mathbb{U} \to \mathbb{C}$ with $h(d) = 0$, $h'(d) = 1$ such that

$$\phi_t(z) = h^{-1}(e^{G'(d)t} h(z)), \quad z \in \mathbb{U}, \quad t \geq 0. \tag{2.5}$$

2) If $d \in \partial\mathbb{U} = \mathbb{R} \cup \{\infty\}$, then there is a unique univalent function $h : \mathbb{U} \to \mathbb{C}$ with $h(i) = 0$, $h'(i) = 1$ such that

$$\phi_t(z) = h^{-1}(h(z) + G(i)t), \quad z \in \mathbb{U}, \quad t \geq 0. \tag{2.6}$$

We call h in either (2.5) or (2.6) the associated univalent function of $\{\phi_t\}$.

Proof. The above derived by the corresponding results about the associated univalent function k of the conjugate semigroup $\{\psi_t\}$ shown in [16, p. 234] and the observation that $b = \gamma^{-1}(d)$ is the corresponding DW point of $\{\psi_t\}$.

Namely if $d \in \mathbb{U}$, i.e. $b \in \mathbb{D}$, then $k(\psi_t(w)) = e^{\widetilde{G}'(b)t}k(w)$, $w \in \mathbb{D}$, with $k(b) = 0$ and $k'(b) = 1$, from which

$$k(\gamma^{-1}(\phi_t(z))) = e^{\widetilde{G}'(b)t}k(\gamma^{-1}(z)), \quad z \in \mathbb{U}.$$

By (2.4) we get $\widetilde{G}'(b) = G'(d) - \frac{2}{d+i}G(d)$ and the representation (2.3) says that $G(d) = 0$. Hence $\widetilde{G}'(b) = G'(d)$ and (2.5) follows by setting $h = \frac{(d+i)^2}{2i} k \circ \gamma^{-1}$.

If $d \in \partial\mathbb{U}$, i.e. $b \in \partial\mathbb{D}$, then $k(\psi_t(w)) = k(w) + \widetilde{G}(0)t$, $w \in \mathbb{D}$, with $k(0) = 0$ and $k'(0) = 1$, from which

$$k(\gamma^{-1}(\phi_t(z))) = k(\gamma^{-1}(z)) + \widetilde{G}(0)t \quad z \in \mathbb{U}.$$

Since by (2.4) $\widetilde{G}(0) = \frac{2i}{(\gamma(0)+i)^2}G(i) = \frac{1}{2i}G(i)$, setting $h = 2ik\circ\gamma^{-1}$ (2.6) follows. □

2.3. **Composition semigroups on $H^p(\mathbb{U})$.** Let $0 < p < \infty$. Given a semigroup $\{\phi_t\}$, the induced semigroup $\{T_t\}$ of composition operators on $H^p(\mathbb{U})$ does not need always to consists of bounded operators, as the following examples shows. However, each T_t is bounded with $\|T_t\| \le 1$ if and only if the DW point of $\{\phi_t\}$ is ∞ (Corollary 2.3).

Example 2.5. Consider the family of analytic functions

$$\phi_t(z) = i\frac{z + i + e^{-t}(z - i)}{z + i - e^{-t}(z - i)}, \quad z \in \mathbb{U}, \quad t \ge 0.$$

For each t, $\phi_t(z) = \gamma(e^{-t}\gamma^{-1}(z))$, that is $\psi_t(w) = e^{-t}w,\ w \in \mathbb{D}$, is the conjugate function of ϕ_t. From this it is clear that each ϕ_t maps $\mathbb{U}$ into $\mathbb{U}$ and that $\{\phi_t\}$ is a semigroup. Also we have that

$$\angle \lim_{z\to\infty} \phi_t(z) = i\frac{1 + e^{-t}}{1 - e^{-t}}, \quad t \ge 0.$$

So for $t > 0$ we get that $\angle \lim_{z\to\infty} \phi_t(z) < \infty$, thus $\phi_t'(\infty) = 0$, which implies that each $T_t,\ t > 0$, is an unbounded operator on $H^p(\mathbb{U})$, $0 < p < \infty$.

Example 2.6. Consider the family of analytic functions

$$\phi_t(z) = (z + 1)^{e^{-t}} - 1, \quad z \in \mathbb{U}, \quad t \ge 0.$$

For each $z \in \mathbb{U}$,

$$\begin{aligned} \mathrm{Im}\phi_t(z) &= \mathrm{Im}e^{e^{-t}(\log|z+1|+iArg(z+1))} \\ &= e^{e^{-t}\log|z+1|}\mathrm{Im}e^{ie^{-t}Arg(z+1)} \\ &= |z+1|^{e^{-t}}\sin(e^{-t}Arg(z+1)) > 0, \end{aligned}$$

so each ϕ_t maps $\mathbb{U}$ into $\mathbb{U}$. Also $\phi_0(z) = z$ and for $t, s \ge 0$

$$\phi_t(\phi_s(z)) = [(z + 1)^{e^{-s}} - 1 + 1]^{e^{-t}} - 1 = \phi_{t+s}(z),$$

thus $\{\phi_t\}$ is a semigroup. Now we have that

$$\phi_t'(z) = e^{-t}(z + 1)^{e^{-t}-1}, \quad t \ge 0.$$

Thus $\phi_t'(\infty) = 0$ for $t > 0$ and so each $T_t,\ t > 0$, is an unbounded operator on $H^p(\mathbb{U})$, $0 < p < \infty$.

Theorem 2.7. *Let $\{\phi_t : t \geq 0\}$ be a semigroup of analytic self-maps of $\mathbb{U}$ and $0 < p < \infty$. Then the following are equivalent:*

(1) *For each $t > 0$ the composition operator $T_t : f \mapsto f \circ \phi_t$ is bounded on $H^p(\mathbb{U})$.*
(2) *There exists $t > 0$ such that T_t is bounded on $H^p(\mathbb{U})$.*
(3) *There exists $t > 0$ such that the angular derivative $\phi_t'(\infty) > 0$.*
(4) *For each $t > 0$ the angular derivative $\phi_t'(\infty) > 0$.*

Moreover if one of the above assertions holds, then

$$\|T_t\| = \phi_1'(\infty)^{-\frac{t}{p}}. \tag{2.7}$$

Proof. We saw in (2.2) that T_t is bounded on $H^p(\mathbb{U})$ if and only if $\phi_t'(\infty) > 0$, in which case

$$\|T_t\| = \phi_t'(\infty)^{-\frac{1}{p}}.$$

Suppose now there exists $\phi_s,\ s > 0$, such that $\phi_s'(\infty) > 0$. Then by Lemma 2.2 we get that for the conjugate function ψ_s on $\mathbb{D}$

$$\psi_s(1) = 1 \text{ and } \psi_s'(1) < \infty.$$

It is known (see [3, Theorems 1 and 5], [5]) that all members of the semigroup $\{\psi_t\}$ have common boundary fixed points, that is $\psi_t(1) = 1$ for each t. Furthermore, since $\psi_s'(1) < \infty$ for some s, [4, Lemmas 1 and 3] say that

$$\psi_t'(1) = \psi_1'(1)^t < \infty \quad \text{for each } t \geq 0,$$

which implies that

$$\phi_t'(\infty) = \phi_1'(\infty)^t > 0$$

for each t and the conclusion follows. □

Moreover the property that $\{T_t\}$ consists of bounded operators depends on the behavior of the infinitesimal generator $G(z)$ of $\{\phi_t\}$ as $z \to \infty$ non-tangentially.

Theorem 2.8. *Let $\{\phi_t : t \geq 0\}$ be a semigroup of analytic self-maps of $\mathbb{U}$ with infinitesimal generator G and $0 < p < \infty$. Then the following are equivalent:*

(1) *Each composition operator $T_t : f \mapsto f \circ \phi_t$ is bounded on $H^p(\mathbb{U})$.*
(2) *The non-tangential limit*

$$\delta := \angle \lim_{z\to\infty} \frac{G(z)}{z}$$

exists finitely.
(3) *The non-tangential limit*

$$\angle \lim_{z\to\infty} G'(z)$$

exists finitely.

Moreover if one of the above assertions holds, then
i) $\delta = \angle \lim_{z\to\infty} G'(z) \in \mathbb{R}$ and
ii) $\|T_t\| = e^{-\frac{\delta t}{p}}$ for each $t \geq 0$.

Proof. $(1 \Leftrightarrow 2)$. This is based on a theorem of M. D. Contreras, S. Díaz-Madrigal and Ch. Pommerenke in [4]. Each T_t is bounded if and only if $\phi_t'(\infty) > 0$, that is if and only if $\psi_t(1) = 1$ and $\psi_t'(1) < \infty$ for each t. The last, by [4, Theorem 1], is

equivalent with the finitely existence of $\angle\lim_{w\to1}\frac{\widetilde{G}(w)}{1-w}$, where $\widetilde{G}$ is the generator of the conjugate semigroup $\{\psi_t\}$. As we shown in relation (2.4),

$$\widetilde{G}(\gamma^{-1}(z)) = \frac{2i}{(z+i)^2}G(z).$$

From this and Lemma 2.1

$$\angle\lim_{w\to1}\frac{\widetilde{G}(w)}{1-w} = \angle\lim_{w\to1}\frac{G(\gamma(w))}{\gamma(w)+i} = \angle\lim_{z\to\infty}\frac{G(z)}{z+i} = \angle\lim_{z\to\infty}\frac{G(z)}{z}$$

and the equivalence follows. Moreover then [4, Theorem 1] implies that $\delta\in\mathbb{R}$ and that $\psi_t'(1) = e^{-\delta t}$, that is $\|T_t\| = e^{-\frac{\delta t}{p}}$ for each $t\geq 0$.

($2\Leftrightarrow 3$). Suppose now δ exists finitely. Then we can write

$$G(z) = \delta z + h(z), \quad z\in\mathbb{U}$$

where

$$\angle\lim_{z\to\infty}\frac{h(z)}{z} = 0. \tag{2.8}$$

Fix $z\in\mathbb{U}$. If r is small enough that $\{z+re^{i\theta}: 0\leq\theta\leq2\pi\}$ lies in $\mathbb{U}$, then by the Cauchy integral formula we have

$$\begin{aligned} G'(z) &= \frac{1}{2\pi}\int_0^{2\pi}\frac{G(z+re^{i\theta})}{re^{i\theta}}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{\delta(z+re^{i\theta})}{re^{i\theta}} + \frac{h(z+re^{i\theta})}{re^{i\theta}}\,d\theta \\ &= \delta + \frac{1}{2\pi}\int_0^{2\pi}\frac{h(z+re^{i\theta})}{z+re^{i\theta}}\,\frac{z+re^{i\theta}}{re^{i\theta}}\,d\theta. \end{aligned}$$

We show that the last integral tends to zero as $z\to\infty$ non-tangentially, thus $\angle\lim_{z\to\infty}G'(z) = \delta$. Fix a non-tangential sector $T_u(\infty)$, $u>0$ and let a sequence $z_n\to\infty$ through $T_u(\infty)$. For each n choose r_n to be the distance of z_n to the boundary of $T_{2u}(\infty)$. It follows from (2.8) that there is $M>0$ such that

$$\Big|\frac{h(z_n+r_ne^{i\theta})}{z_n+r_ne^{i\theta}}\Big| < M, \quad \text{for all } n \text{ and } \theta.$$

Furthermore, let ω be the smallest angle made by the boundary lines of $T_u(\infty)$ and $T_{2u}(\infty)$, then $\frac{r_n}{|z_n|} > \sin\omega > 0$ for each n and so

$$\Big|\frac{z_n+r_ne^{i\theta}}{r_ne^{i\theta}}\Big| < \frac{|z_n|}{r_n}+1 < \frac{1}{\sin\omega}+1.$$

Therefore an application of Lebesgue's dominated convergence theorem and (2.8) gives that $G'(z_n)\to\delta$. Since u and $\{z_n\}$ are arbitrary our conclusion follows.

Conversely suppose $\angle\lim_{z\to\infty}G'(z)$ exists finitely. Fix $u>0$ and let $z_n\to\infty$ through the sector $T_u(\infty)$. We can suppose that $\mathrm{Im}z_n>1$ for each n and we can write

$$\begin{aligned} \frac{G(z_n)-G(i)}{z_n-i} &= \frac{1}{z_n-i}\int_0^1\frac{\partial}{\partial t}\Big(G((z_n-i)t+i)\Big)\,dt \\ &= \int_0^1 G'((z_n-i)t+i)\,dt. \end{aligned}$$

Since $\angle \lim_{z\to\infty} G'(z) < \infty$, there is $M > 0$ such that $|G'(z_n)| < M$ for all n and applying Lebesgue's dominated convergence theorem we get

$$\lim_{n\to\infty} \frac{G(z_n) - G(i)}{z_n - i} = \int_0^1 \lim_{n\to\infty} G'((z_n - i)t + i)\, dt = \angle \lim_{z\to\infty} G'(z).$$

Furthermore

$$\begin{aligned}\lim_{n\to\infty} \frac{G(z_n) - G(i)}{z_n - i} &= \lim_{n\to\infty} \frac{G(z_n) - G(i)}{z_n - i} + \lim_{n\to\infty} \frac{G(i)}{z_n - i} \\ &= \lim_{n\to\infty} \frac{G(z_n)}{z_n - i} = \lim_{n\to\infty} \frac{G(z_n)}{z_n - i}\frac{z_n - i}{z_n} \\ &= \lim_{n\to\infty} \frac{G(z_n)}{z_n}.\end{aligned}$$

Since u and $\{z_n\}$ are arbitrary we get that $\angle \lim_{z\to\infty} \frac{G(z)}{z} = \angle \lim_{z\to\infty} G'(z) < \infty$, completing the proof. □

3. Strong continuity and infinitesimal generator

Throughout this section $\{\phi_t\}$ is a semigroup which induces a semigroup $\{T_t\}$ of bounded composition operators on $H^p(\mathbb{U})$ spaces.

Lemma 3.1. *Let $0 < p < \infty$ and $\lambda \in \mathbb{C}$, then $h_\lambda(z) = (z + i)^\lambda \in H^p(\mathbb{U})$ if and only if $Re\lambda < -\frac{1}{p}$.*

Proof. Choosing a logarithmic branch we have $|(z+i)^\lambda| = e^{\mathrm{Re}\lambda \log|z+i| - \mathrm{Im}\lambda \arg(z+i)}$ for each $z \in \mathbb{U}$ and we can find constants $c, c' > 0$ such that

$$c'|z + i|^{\mathrm{Re}\lambda} \leq |(z + i)^\lambda| \leq c|z + i|^{\mathrm{Re}\lambda}.$$

Thus we can suppose that λ is real. Then

$$\begin{aligned}\|h_\lambda\|_p^p &= \sup_{y>0} \int_{-\infty}^{+\infty} \left(\frac{1}{x^2 + (1+y)^2}\right)^{-\frac{\lambda p}{2}} dx \\ &= \sup_{y>0} (1+y)^{1+p\lambda} \int_{-\infty}^{+\infty} \left(\frac{1}{x^2+1}\right)^{-\frac{\lambda p}{2}} dx.\end{aligned}$$

The last integral is convergent if and only if $-\frac{\lambda p}{2} > \frac{1}{2}$, i.e. $\lambda < -\frac{1}{p}$, giving our conclusion. □

The growth condition of $H^p(\mathbb{U})$ functions in the following lemma is known, see for instance [7, p. 188], [11, p. 53]. Here we find the best possible constant.

Lemma 3.2. *Let $0 < p < \infty$ and suppose $f \in H^p(\mathbb{U})$. Then for each $z \in \mathbb{U}$,*

$$|f(z)|^p \leq \frac{1}{4\pi} \frac{\|f\|_p^p}{Im z}, \tag{3.1}$$

where the constant $\frac{1}{4\pi}$ is the best possible.

Proof. We examine first the case $p = 2$. We consider a point $z \in \mathbb{U}$ and we recall that the $H^2(\mathbb{U})$ function

$$k_z(w) = \frac{i}{2\pi(w - \overline{z})}, \quad w \in \mathbb{U},$$

is the reproducing kernel of $H^2(\mathbb{U})$, that is for each $f \in H^2(\mathbb{U})$

$$f(z) = \langle f, k_z \rangle,$$

where the pairing is the inner product of $H^2(\mathbb{U})$. From this

$$\|k_z\|_2^2 = \langle k_z, k_z \rangle = k_z(z) = \frac{1}{4\pi \mathrm{Im} z}.$$

Let $f \in H^2(\mathbb{U})$. Then Cauchy-Schwarz inequality implies that

$$|f(z)|^2 = |\langle f, k_z \rangle|^2 \leq \|f\|_2^2 \|k_z\|_2^2 = \frac{\|f\|_2^2}{4\pi \mathrm{Im} z}.$$

For $p \neq 2$, let $f \in H^p(\mathbb{U})$ ($f \not\equiv 0$). By the factorization of functions in $H^p(\mathbb{U})$ (see [7, Theorem 11.3])

$$f(z) = b(z)g(z), \qquad z \in \mathbb{U},$$

where g is a nonvanishing $H^p(\mathbb{U})$ function such that

$$|f(x)| = |g(x)| \ \text{ almost everywhere on } \mathbb{R}$$

and b is a Blaschke product for the upper half plane of the form

$$b(z) = \Big(\frac{z-i}{z+i}\Big)^m \prod_n \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\overline{z_n}},$$

where m is a nonnegative integer and z_n are the zeros ($z_n \neq i$) of f in $\mathbb{U}$. Since $\left|\frac{z-a}{z-\overline{a}}\right| < 1$ for each $z, a \in \mathbb{U}$, it follows that $|b(z)| < 1$, thus

$$|f(z)| \leq |g(z)| \ \text{ for each } z \in \mathbb{U}.$$

Moreover, since g is a nonvanishing $H^p(\mathbb{U})$ function, we can choose a single-valued branch of $g^{p/2}$, which belongs to $H^2(\mathbb{U})$ and by the case $p = 2$ follows that

$$\begin{aligned} |f(z)|^p \leq |g^{p/2}(z)|^2 &\leq \frac{\|g^{p/2}\|_2^2}{4\pi \mathrm{Im} z} \\ &= \frac{1}{4\pi \mathrm{Im} z} \int_{-\infty}^{\infty} |g(x)|^p \, dx \\ &= \frac{1}{4\pi \mathrm{Im} z} \int_{-\infty}^{\infty} |f(x)|^p \, dx \\ &= \frac{1}{4\pi} \frac{\|f\|_p^p}{\mathrm{Im} z}. \end{aligned}$$

Finally, for each $0 < p < \infty$ considering the estimate for the $H^p(\mathbb{U})$ function $h_{-2/p}(z) = (z+i)^{-\frac{2}{p}}$, for which $\|h_{-2/p}\|_p^p = \pi$, we see that at $z = i$ equality holds, which implies that $\frac{1}{4\pi}$ is the best possible constant for this inequality. $\square$

Theorem 3.3. *Suppose $1 \leq p < \infty$ and let $\{\phi_t\}$ be a semigroup of analytic self-maps of $\mathbb{U}$ which induces a semigroup $\{T_t\}$ of bounded composition operators on $H^p(\mathbb{U})$. Then*

(1) *$\{T_t\}$ is strongly continuous on $H^p(\mathbb{U})$.*
(2) *If G is the generator of $\{\phi_t\}$, then the infinitesimal generator Γ of $\{T_t\}$ has domain of definition*

$$D(\Gamma) = \{f \in H^p(\mathbb{U}) : Gf' \in H^p(\mathbb{U})\}$$

and is given by

$$\Gamma(f) = Gf', \quad f \in D(\Gamma).$$

Proof. (1) For the strong continuity we need to show

$$\lim_{t\to 0}\|T_t(f)-f\|_p=0,$$

for every $f\in H^p(\mathbb{U})$. Fix a function $f\in H^p(\mathbb{U})$. Since the set of $H^p(\mathbb{U})$ functions that are continuous on $\mathbb{U}\cup\mathbb{R}$, denoted by $\mathcal{A}^p(\mathbb{U})$, is dense in $H^p(\mathbb{U})$ (see [11, Corollary 3.3]), for arbitrary $\epsilon>0$ we can find $g\in\mathcal{A}^p(\mathbb{U})$ such that $\|f-g\|_p<\epsilon$. Then

$$\begin{aligned}\|T_t(f)-f\|_p &\le \|T_t(f)-T_t(g)\|_p+\|T_t(g)-g\|_p+\|g-f\|_p\\ &\le (\|T_t\|+1)\|f-g\|_p+\|T_t(g)-g\|_p\end{aligned}$$

and further by Theorem 2.7 follows that

$$\|T_t(f)-f\|_p\le(\phi_1'(\infty)^{-\frac{t}{p}}+1)\epsilon+\|T_t(g)-g\|_p.$$

Therefore, since $\phi_1'(\infty)^{-\frac{t}{p}}$ as a function of t is uniformly bounded on bounded intervals of $[0,\ +\infty)$, we see that it suffices to show that for each $g\in\mathcal{A}^p(\mathbb{U})$

$$\|T_t(g)-g\|_p=\|g\circ\phi_t-g\|_p\to 0,\qquad \text{as } t\to 0.$$

By way of contradiction, suppose there is a function $g\in\mathcal{A}^p(\mathbb{U})$ and a sequence $\{t_n\}$ of values of t such that $t_n\to 0$ as $n\to\infty$ and

$$\|g\circ\phi_{t_n}-g\|_p^p=\int_{-\infty}^{\infty}|g(\phi_{t_n}(x))-g(x)|^p\,dx\ge s>0,\qquad \text{for each } n.$$

We will show first that there is a subsequence $\{t_{n_k}\}$ such that

$$g(\phi_{t_{n_k}}(x))\to g(x) \text{ almost everywhere on } \mathbb{R}. \tag{3.2}$$

To do this we consider the $H^2(\mathbb{U})$ function $h(z)=\pi^{-\frac{1}{2}}(z+i)^{-1}$ for which

$$\|h\|_2^2=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx=1$$

and we will show that $\|h\circ\phi_{t_n}-h\|_2\to 0$. If this is true, then $h\circ\phi_{t_n}\to h$ in measure on $\mathbb{R}$ and thus there is a subsequence $\{t_{n_k}\}$ such that $h(\phi_{t_{n_k}}(x))\to h(x)$ almost everywhere on $\mathbb{R}$ (see for instance [17, Ex. 9, p. 85]), from which

$$\phi_{t_{n_k}}(x)\to x,\qquad \text{a. e. on } \mathbb{R}.$$

Hence, since g is continuous on $\mathbb{U}\cup\mathbb{R}$, (3.2) follows. Now to show that $\|h\circ\phi_{t_n}-h\|_2\to 0$ we use the parallelogram law which asserts that

$$\|h\circ\phi_{t_n}-h\|_2^2+\|h\circ\phi_{t_n}+h\|_2^2=2(\|h\circ\phi_{t_n}\|_2^2+\|h\|_2^2)$$

for each n. From this and the norm of each T_t in Theorem 2.7 follows

$$\begin{aligned}\|h\circ\phi_{t_n}-h\|_2^2&\le 2(\phi_1'(\infty)^{-t_n}+1)\|h\|_2^2-\|h\circ\phi_{t_n}+h\|_2^2\\ &=2(\phi_1'(\infty)^{-t_n}+1)-\|h\circ\phi_{t_n}+h\|_2^2.\end{aligned}$$

Further from the growth estimate (3.1)

$$\|h\circ\phi_{t_n}+h\|_2^2\ge|h(\phi_{t_n}(i))+h(i)|^2 4\pi,$$

thus

$$\|h\circ\phi_{t_n}-h\|_2^2\le 2(\phi_1'(\infty)^{-t_n}+1)-|h(\phi_{t_n}(i))+h(i)|^2 4\pi.$$

Since $2(\phi_1'(\infty)^{-t_n}+1)\to 4$ and $|h(\phi_{t_n}(i))+h(i)|^2 4\pi\to 4|h(i)|^2 4\pi=4$ as $n\to\infty$, we get that

$$\|h\circ\phi_{t_n}-h\|_2\to 0.$$

Next we consider the sequence of functions

$$S_{t_{n_k}}(x) = 2^p\big(|g(\phi_{t_{n_k}}(x))|^p + |g(x)|^p\big) - |g(\phi_{t_{n_k}}(x)) - g(x)|^p$$

defined for almost all $x \in \mathbb{R}$. A standard inequality, $(a+b)^p \leq 2^p(a^p+b^p)$, $a, b \geq 0$, shows that the above functions are nonnegative. Since as found above

$$g(\phi_{t_{n_k}}(x)) \to g(x) \text{ almost everywhere on } \mathbb{R},$$

an application of Fatou's lemma to the sequence $\{S_{t_{n_k}}\}$ gives

$$\begin{aligned}
2^{p+1}\|g\|_p^p &\leq \liminf_{k\to\infty} \int_{-\infty}^{\infty} 2^p\big(|g(\phi_{t_{n_k}}(x))|^p + |g(x)|^p\big) - |g(\phi_{t_{n_k}}(x)) - g(x)|^p \, dx \\
&= \liminf_{k\to\infty} \Big[2^p\Big(\int_{-\infty}^{\infty} |g(\phi_{t_{n_k}}(x))|^p \, dx + \int_{-\infty}^{\infty} |g(x)|^p \, dx\Big) \\
&\qquad\qquad - \int_{-\infty}^{\infty} |g(\phi_{t_{n_k}}(x)) - g(x)|^p \, dx\Big] \\
&\leq \liminf_{k\to\infty} \Big[2^p(\phi_1'(\infty)^{-t_{n_k}} + 1)\|g\|_p^p - \int_{-\infty}^{\infty} |g(\phi_{t_{n_k}}(x)) - g(x)|^p \, dx\Big] \\
&= 2^{p+1}\|g\|_p^p - \limsup_{k\to\infty} \int_{-\infty}^{\infty} |g(\phi_{t_{n_k}}(x)) - g(x)|^p \, dx.
\end{aligned}$$

Thus

$$0 \geq \limsup_{k\to\infty} \int_{-\infty}^{\infty} |g(\phi_{t_{n_k}}(x)) - g(x)|^p \, dx \geq \liminf_{k\to\infty} \int_{-\infty}^{\infty} |g(\phi_{t_{n_k}}(x)) - g(x)|^p \, dx \geq 0$$

and so we conclude that

$$\|g \circ \phi_{t_{n_k}} - g\|_p \to 0,$$

which contradicts the original choice of g and $\{t_n\}$.

(2) By definition the domain $D(\Gamma)$ of Γ consists of all $f \in H^p(\mathbb{U})$ for which the limit $\lim_{t\to 0} \frac{T_t(f)-f}{t}$ exists in $H^p(\mathbb{U})$ and

$$\Gamma(f) = \lim_{t\to 0} \frac{T_t(f) - f}{t}, \quad f \in D(\Gamma).$$

The growth estimate (3.1) shows that convergence in the norm of $H^p(\mathbb{U})$ implies in particular pointwise convergence, therefore for $f \in D(\Gamma)$,

$$\begin{aligned}
\Gamma(f)(z) = \lim_{t\to 0} \frac{T_t(f)(z) - f(z)}{t} &= \frac{\partial f(\phi_t(z))}{\partial t}\Big|_{t=0} \\
&= f'(z)\frac{\partial \phi_t(z)}{\partial t}\Big|_{t=0} = G(z)f'(z).
\end{aligned}$$

This shows that $D(\Gamma) \subseteq \{f \in H^p(\mathbb{U}) : Gf' \in H^p(\mathbb{U})\}$. Conversely let $f \in H^p(\mathbb{U})$ such that $Gf' \in H^p(\mathbb{U})$. Then for $z \in \mathbb{U}$,

$$\begin{aligned}
T_t(f)(z) - f(z) &= \int_0^t \frac{\partial f(\phi_s(z))}{\partial s} \, ds \\
&= \int_0^t \frac{\partial \phi_s(z)}{\partial s} f'(\phi_s(z)) \, ds \\
&= \int_0^t G(\phi_s(z)) f'(\phi_s(z)) \, ds.
\end{aligned}$$

Therefore

$$\frac{T_t(f)-f}{t}=\frac{1}{t}\int_0^t T_s(Gf')\,ds.$$

Since $\{T_t\}$ is strongly continuous the latter tends in the norm of $H^p(\mathbb{U})$ to Gf' as $t\to 0$ (see [14, Theorem 2.4, p. 4]). Thus $f\in D(\Gamma)$, completing the proof. □

Corollary 3.4. *Suppose $1\le p<\infty$. The only uniformly continuous semigroup $\{T_t\}$ on $H^p(\mathbb{U})$ is the one induced by the trivial semigroup.*

Proof. Suppose a semigroup $\{\phi_t\}$ with generator G induces a semigroup $\{T_t\}$ which is continuous in the uniform operator topology. Then the infinitesimal generator Γ of $\{T_t\}$ is bounded on $H^p(\mathbb{U})$. Thus for each $f\in H^p(\mathbb{U})$ by Theorem 3.3 we have that $\Gamma(f)=Gf'\in H^p(\mathbb{U})$ and moreover

$$\|Gf'\|_p\le\|\Gamma\|\|f\|_p.$$

Consider now for each n natural the analytic functions

$$e_n(z)=\frac{1}{\pi^{1/p}}\frac{(\gamma^{-1}(z))^n}{(z+i)^{2/p}},\quad z\in\mathbb{U},$$

where we recall that $\gamma^{-1}(z)=\frac{z-i}{z+i}:\mathbb{U}\to\mathbb{D}$. By Lemma 3.1 we see that $e_n\in H^p(\mathbb{U})$ and furthermore

$$\|e_n\|_p^p=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{|\gamma^{-1}(x)|^{pn}}{|x+i|^2}\,dx=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{1}{x^2+1}\,dx=1.$$

Thus $\|Ge_n'\|_p\le\|\Gamma\|<\infty$ for each n. A short computation gives

$$e_n'(z)=\frac{1}{\pi^{1/p}}\frac{(\gamma^{-1}(z))^{n-1}}{(z+i)^{\frac{2}{p}+2}}\big[-\frac{2}{p}z+(2n+\frac{2}{p})i\big].$$

Also let $\omega(z)=-\frac{p}{p+2}(z+i)^{-\frac{2}{p}-1}$ for which $\omega'(z)=(z+i)^{-\frac{2}{p}-2}$. Since from Lemma 3.1 $\omega\in H^p(\mathbb{U})$ we get that $G\omega'\in H^p(\mathbb{U})$. Therefore

$$\begin{aligned}\|Ge_n'\|_p^p&=\frac{1}{\pi}\int_{-\infty}^{\infty}|(G\omega')(x)|^p|(\gamma^{-1}(x))^{n-1}[-\frac{2}{p}x+(2n+\frac{2}{p})i]|^p\,dx\\&\ge\frac{n^p}{\pi}\int_{-\infty}^{\infty}|(G\omega')(x)|^p\,dx=\frac{n^p}{\pi}\|G\omega'\|_p^p\end{aligned}$$

and it follows that for each n, $\frac{n}{\pi^{1/p}}\|G\omega'\|_p\le\|\Gamma\|<\infty$. Thus we conclude that $G\equiv 0$, that is $\{\phi_t\}$ is trivial, which obviously induces an uniformly continuous semigroup $\{T_t\}$. □

Proposition 3.5. *Suppose $1\le p<\infty$ and let $\{\phi_t\}$ be a semigroup which induces a semigroup $\{T_t\}$ of bounded operators on $H^p(\mathbb{U})$. If G is the generator, d the DW point and h the associated univalent function of $\{\phi_t\}$, then we have the following for $\sigma_\pi(\Gamma)$, the point spectrum of the generator Γ of $\{T_t\}$.*
i) If $d\in\mathbb{U}$, then

$$\sigma_\pi(\Gamma)=\{G'(d)k: h(z)^k\in H^p(\mathbb{U}),\ k=0,1,2,...\}.$$

ii) If $d\in\partial\mathbb{U}$, then

$$\sigma_\pi(\Gamma)=\{G(i)\nu\in\mathbb{C}: e^{\nu h(z)}\in H^p(\mathbb{U})\}.$$

Proof. We have to solve $\Gamma(f) = \lambda f$ for $\lambda \in \mathbb{C}$ and $f \in D(\Gamma)$, $f \not\equiv 0$. This by Theorem 3.3 is equivalent to the differential equation

$$G(z)f'(z) = \lambda f(z), \quad f \in H^p(\mathbb{U}), \quad f \not\equiv 0. \tag{3.3}$$

i) Suppose $d \in \mathbb{U}$. Then $\phi_t(z) = h^{-1}(e^{G'(d)t}h(z))$ for each t with $h(d) = 0$ and $h'(d) = 1$ (see (2.5)). So we get that

$$G(z) = \frac{\partial \phi_t(z)}{\partial t}\Big|_{t=0} = G'(d)\frac{e^{G'(d)t}h(z)}{h'(\phi_t(z))}\Big|_{t=0} = G'(d)\frac{h(z)}{h'(z)}.$$

Notice that $G'(d) \neq 0$, since otherwise $\{\phi_t\}$ will be trivial. Thus (3.3) becomes

$$\frac{h(z)}{h'(z)}f'(z) = \frac{\lambda}{G'(d)}f(z), \quad f \in H^p(\mathbb{U}), \quad f \not\equiv 0.$$

If a $\lambda \in \mathbb{C}$ and a function f satisfy this, choosing $r \in (0,\ \mathrm{Im} d)$ such that $f(z)$ has no zeros on $|z - d| = r$, we get

$$\frac{1}{2\pi i}\int_{|z-d|=r} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \frac{\lambda}{G'(d)}\frac{1}{2\pi i}\int_{|z-d|=r} \frac{h'(\zeta)}{h(\zeta)}\, d\zeta$$

and by the argument principle follows that $\lambda = G'(d)k$, where k is a nonnegative integer. Since the nonzero analytic solutions of

$$\frac{h(z)}{h'(z)}f'(z) = kf(z)$$

are of the form $ch(z)^k,\ c \neq 0$, the conclusion follows.

ii) Suppose $d \in \partial\mathbb{U}$. Then $\phi_t(z) = h^{-1}(h(z) + G(i)t)$ for each t (see (2.6)) and

$$G(z) = \frac{\partial \phi_t(z)}{\partial t}\Big|_{t=0} = \frac{G(i)}{h'(\phi_t(z))}\Big|_{t=0} = \frac{G(i)}{h'(z)},$$

where, since $\{\phi_t\}$ is not trivial, $G(i) \neq 0$. Thus by (3.3) we see that $G(i)\nu \in \sigma_\pi(\Gamma)$ if and only if there exists a function $f \in H^p(\mathbb{U})$, $f \not\equiv 0$, that satisfies

$$\frac{f'(z)}{h'(z)} = \nu f(z).$$

Since the nonzero analytic solutions of this are of the form $ce^{\nu h(z)},\ c \neq 0$, the conclusion follows. $\square$

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